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AVERAGES OVER SPHERES FOR KINETIC TRANSPORT EQUATIONS; HYPERBOLIC SOBOLEV SPACES AND STRICHARTZ INEQUALITIES

Nikolaos BOURNAVEAS [a], **Benoit PERTHAME** [b]

[a] *Department of Mathematics and Statistics, University of Edinburgh, JCMB, The King's Buildings, Edinburgh EH9 3JZ, UK*

[b] *Ecole Normale Supérieure, Département de Mathématiques et Applications, 45 rue d' Ulm, 75230 Paris Cedex 05, France*

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ABSTRACT. – We consider averages over spheres for kinetic transport equations in two space dimensions. In this case, $1/4$ derivative is lost in the various forms of the averaging lemmas. We show that it is possible to recover the optimal regularity working in the hyperbolic Sobolev spaces. Strichartz type inequalities follow with better exponents than those given by classical Sobolev imbeddings. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

In this paper we consider averages over spheres

$$\rho_S(t,x) = \int_{S^{d-1}} f(t,x,v)\,\mathrm{d}\sigma(v), \tag{1}$$

where $f:\mathbb{R}\times\mathbb{R}^d\times S^{d-1}\to\mathbb{R}$ is a solution of a kinetic transport equation

$$\partial_t f + v\cdot\nabla_x f = g. \tag{2}$$

We prove that, like in the case of averaging over balls, this averaged quantity is more regular than f itself but, specifically to the sphere, we prove that optimal regularity can be proved in the so-called *hyperbolic Sobolev spaces*. The regularizing effect in classical Sobolev spaces was first discovered in [12] and [11]. The theory of averaging lemmas was then developed and proved to be optimal in several papers (see for instance [7–10,15,16,6,24,3]). See also [2] and the references therein. They play a fundamental role for proving compactness, and thus existence, not only

E-mail addresses: nikolaos@maths.ed.ac.uk (N. Bournaveas), benoit.perthame@ens.fr (B. Perthame).

in kinetic equations but also in several nonlinear Partial Differential Equations or variational problems. Strichartz type inequalities were discovered for (2) in [5] and improved in [13].

In the simplest case, namely for $f:\mathbb{R}^{1+2d}\to\mathbb{R}$ a solution of (2) and

$$\rho_B(t,x)=\int\limits_{|v|\leqslant 1} f(t,x,v)\,\mathrm{d}v, \tag{3}$$

the averaging lemma says that ρ_B is smoother than f by half a derivative. More precicely the following estimate holds true:

$$\big\|\big(1+|\tau|+|\xi|\big)^{1/2}\widehat{\rho_B}(\tau,\xi)\big\|_{L^2(\mathbb{R}^{1+d})}\leqslant C(d)\big(\|f\|_{L^2(\mathbb{R}^{1+d}\times B_1)}+\|g\|_{L^2(\mathbb{R}^{1+d}\times B_1)}\big), \tag{4}$$

where $B_1:=\{v\in\mathbb{R}^d\colon |v|\leqslant 1\}$. This result is known to be sharp ([16]).

We recall in Section 2 the proof of this estimate for ρ_S when the dimension d is at least 3. In Section 3 we shall consider the same problem in two space dimensions. Due to scaling properties which are particular to two dimensions, the averages on the sphere ρ_S is smoother than f only by $1/4$ derivatives. More precicely

$$\big\|\big(1+|\tau|+|\xi|\big)^{1/4}\widehat{\rho_S}(\tau,\xi)\big\|_{L^2(\mathbb{R}^{1+2})}\leqslant C\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^1)}+\|g\|_{L^2(\mathbb{R}^{1+2}\times S^1)}\big). \tag{5}$$

However, a more careful analysis will show that away from the light cone $|\tau|=|\xi|$ the average ρ_S does have $1/2$ derivatives in L^2. More precicely, we shall prove:

$$\begin{aligned}&\big\|\big(1+\big||\tau|-|\xi|\big|\big)^{1/4}\big(1+|\tau|+|\xi|\big)^{1/4}\widehat{\rho_S}(\tau,\xi)\big\|_{L^2(\mathbb{R}^{1+2})}\\&\quad\leqslant C\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^1)}+\|g\|_{L^2(\mathbb{R}^{1+2}\times S^1)}\big).\end{aligned}$$

Weighted Sobolev spaces of this type, that we will refer to as *hyperbolic Sobolev spaces*, have been used extensively in Schrödinger, KdV and wave equations ([4,17–23,27,14,1] and the references therein). Using one of their embedding properties (see [26]) we shall prove the following Strichartz-type estimate for ρ_S in two dimensions

$$\|\rho_S\|_{L^3(\mathbb{R}^{1+2})}\leqslant C\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^1)}+\|g\|_{L^2(\mathbb{R}^{1+2}\times S^1)}\big).$$

In Section 4 we shall consider equations which contain a v-derivative in the right-hand side, and in Section 5 averages over spheres for solutions of the initial value problem for the homogeneous equation.

NOTATION. – Throughout the paper we define the weights w_+ and w_- by:

$$w_+(\tau,\xi)=1+|\tau|+|\xi|,\qquad w_-(\tau,\xi)=1+\big||\tau|-|\xi|\big|.$$

We shall also use the homogeneous versions

$$\dot w_+(\tau,\xi)=|\tau|+|\xi|,\qquad \dot w_-(\tau,\xi)=\big||\tau|-|\xi|\big|.$$

For simplicity we shall always assume that $f,g,h,\ldots$ are smooth functions which decay sufficiently fast at infinity.

2. Averages over spheres in dimension $d \geqslant 3$

For space dimensions $d \geqslant 3$, averages over spheres gain as much regularity as averages over balls. This is the content of the following theorem:

THEOREM 1 [11]. – *Consider the kinetic-transport equation* (2) *in* $d \geqslant 3$ *dimensions. Then, the averages over spheres in* (1) *satisfy*

$$\left\| w_+^{1/2} \widehat{\rho_S} \right\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \leqslant C(d)\left(\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times S^{d-1})} + \|g\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times S^{d-1})}\right). \tag{6}$$

Proof. – We shall actually prove the following pointwise estimate:

$$\left| w_+(\tau,\xi)^{1/2} \widehat{\rho_S}(\tau,\xi)\right| \leqslant C(d)\left(\left\| \hat f(\tau,\xi,\cdot)\right\|_{L^2(S^{d-1})} + \left\| \hat g(\tau,\xi,\cdot)\right\|_{L^2(S^{d-1})}\right). \tag{7}$$

Since $w_+ = 1 + \dot w_+$ it suffices to prove the following two estimates:

$$\left| \widehat{\rho_S}(\tau,\xi)\right| \leqslant C(d)\left\| \hat f(\tau,\xi,\cdot)\right\|_{L^2(S^{d-1})}, \tag{8}$$

$$\left| \dot w_+(\tau,\xi)^{1/2} \widehat{\rho_S}(\tau,\xi)\right| \leqslant C(d)\left(\left\| \hat f(\tau,\xi,\cdot)\right\|_{L^2(S^{d-1})} + \left\| \hat g(\tau,\xi,\cdot)\right\|_{L^2(S^{d-1})}\right). \tag{9}$$

The first one follows trivially from the definition of ρ_S. To prove the second, add f to both sides of (2) and then take the Fourier transform with respect to t and x. Setting $h = f + g$, we obtain

$$\hat f(\tau,\xi,v) = \frac{\hat h(\tau,\xi,v)}{1 + \mathrm{i}(\tau + v\cdot\xi)},$$

$$\widehat{\rho_S}(\tau,\xi) = \int_{S^{d-1}} \frac{\hat h(\tau,\xi,v)}{1 + \mathrm{i}(\tau + v\cdot\xi)}\,\mathrm{d}\sigma(v). \tag{10}$$

Using Cauchy–Schwarz inequality we deduce

$$\left| \widehat{\rho_S}(\tau,\xi)\right| \leqslant J(\tau,\xi)\left(\int_{S^{d-1}} \left|\hat h(\tau,\xi,v)\right|^2 \mathrm{d}\sigma(v)\right)^{1/2}, \tag{11}$$

where

$$J(\tau,\xi) = \left(\int_{S^{d-1}} \frac{1}{1 + (\tau + v\cdot\xi)^2}\,\mathrm{d}\sigma(v)\right)^{1/2}.$$

Therefore, in order to prove (9), it suffices to show that

$$J(\tau,\xi) \leqslant \frac{C(d)}{(|\tau| + |\xi|)^{1/2}}. \tag{12}$$

We may consider a basis in the v space such that $\xi = (0,\dots,0,|\xi|)$. Using spherical coordinates for v we get

$$J^2(\tau,\xi) \leqslant C(d)\int_0^\pi \frac{(\sin\theta)^{d-2}}{1 + (\tau + |\xi|\cos\theta)^2}\,\mathrm{d}\theta.$$

Since $d-2 \geqslant 1$ we have $(\sin\theta)^{d-2} \leqslant \sin\theta$. Therefore

$$J^2(\tau,\xi) \leqslant C(d) \int_0^{\pi} \frac{\sin\theta}{1+(\tau+|\xi|\cos\theta)^2}\,\mathrm{d}\theta.$$

Changing variables $\theta \to x := \tau + |\xi|\cos\theta$, we get

$$J^2(\tau,\xi) \leqslant \frac{C(d)}{|\xi|} \int_{\tau-|\xi|}^{\tau+|\xi|} \frac{\mathrm{d}x}{1+x^2} \tag{13}$$

$$\leqslant \frac{C(d)}{|\xi|}. \tag{14}$$

This proves (12) in the case $|\xi| \geqslant |\tau|$. We now consider the case $|\xi| < \tau$ (the case $\tau < -|\xi|$ is treated similarly). Using (13) we get

$$\begin{aligned} J^2(\tau,\xi) &\leqslant \frac{C(d)}{|\xi|} \int_{\tau-|\xi|}^{\tau+|\xi|} \frac{1}{(1+x)^2}\,\mathrm{d}x \\ &\leqslant \frac{C(d)}{|\xi|}\left(\frac{1}{1+\tau-|\xi|} - \frac{1}{1+\tau+|\xi|}\right) \\ &\leqslant \frac{C(d)}{(1+\tau-|\xi|)(1+\tau+|\xi|)} \qquad (15) \\ &\leqslant \frac{C(d)}{\tau}. \qquad (16) \end{aligned}$$

Hence, we have proved that

$$\int_0^{\pi} \frac{\sin\theta}{1+(\tau+|\xi|\cos\theta)^2}\,\mathrm{d}\theta \leqslant \frac{C(d)}{|\tau|+|\xi|}, \tag{17}$$

and this completes the proof. □

Remarks. – (1) Estimate (6) implies the classical averaging lemma (4). Indeed, using a scaling argument, we can show that the integral $\rho_{S_r}(t,x) := \int_{|v|=r} f(t,x,v)\,\mathrm{d}\sigma(v)$, satisfies

$$\begin{aligned} &\big(1+|\tau|+r|\xi|\big)^{1/2}\big|\widehat{\rho_{S_r}}(\tau,\xi)\big| \\ &\quad \leqslant C(d)\, r^{d-1/2}\big(\big\|\hat f(\tau,\xi,\cdot)\big\|_{L^2(S_r^{d-1})} + \big\|\hat g(\tau,\xi,\cdot)\big\|_{L^2(S_r^{d-1})}\big), \qquad (18) \end{aligned}$$

where $S_r^{d-1} = \{v \in \mathbb{R}^d\colon |v| = r\}$. Using (18) together with $\rho_B(t,x) = \int_0^1 \rho_{S_r}(t,x)\,\mathrm{d}r$ we get

$$\big(1+|\tau|+|\xi|\big)^{1/2}\big|\widehat{\rho_B}(\tau,\xi)\big| \leqslant C(d)\big(\big\|\hat f(\tau,\xi,\cdot)\big\|_{L^2(B_1)} + \big\|\hat g(\tau,\xi,\cdot)\big\|_{L^2(B_1)}\big).$$

Taking the L^2 norm with respect to (τ,ξ) we get (4).

(2) The exponent $1/2$ in (7) is best possible. Because, if (7) was true with an exponent $s > 1/2$ then, repeating the argument above, we would get the classical averaging lemma (4)

with exponent $s > 1/2$. However, it is known from [16] that $1/2$ is the best possible exponent for (4).

(3) Observe that in the "elliptic" region $|\tau| > |\xi|$ we have proven that a stronger estimate holds true, namely:

$$(19) \qquad \bigl(1+|\tau|-|\xi|\bigr)^{1/2}\bigl(1+|\tau|+|\xi|\bigr)^{1/2}\bigl|\widehat{\rho_S}(\tau,\xi)\bigr| \leqslant C(d)\bigl\|\hat{h}(\tau,\xi,\cdot)\bigr\|_{L^2(S^{d-1})}.$$

To get (19) combine (15) with (11). Working similarly we can prove that this stronger estimate is also true for ρ_B, i.e., for $|\tau| > |\xi|$, we have:

$$(20) \qquad \bigl(1+|\tau|-|\xi|\bigr)^{1/2}\bigl(1+|\tau|+|\xi|\bigr)^{1/2}\bigl|\widehat{\rho_B}(\tau,\xi)\bigr| \leqslant C(d)\bigl\|\hat{h}(\tau,\xi,\cdot)\bigr\|_{L^2(B)}.$$

3. Averages over spheres in dimension $d = 2$

In this section we consider averages over spheres in two dimensions. We shall see that these averages gain only $1/4$ derivatives and that the "loss" of $1/4$ derivatives occurs close to the light cone $|\tau| = |\xi|$.

To understand why this happens let us compare $d = 2$ to $d = 3$, from the point of view of scaling. Going back to (17), setting $\theta = \pi - \phi$, and considering the case $\tau = |\xi|$, we see that what was actually estimated there was the integral $\int_0^\pi \frac{\sin\phi\,\mathrm{d}\phi}{1+|\xi|^2(1-\cos\phi)^2}$. For $\phi \simeq 0$ we have

$$\frac{\sin\phi\,\mathrm{d}\phi}{1+|\xi|^2(1-\cos\phi)^2} \simeq \frac{\phi\,\mathrm{d}\phi}{1+|\xi|^2\phi^4} = \frac{1}{|\xi|}\frac{x\,\mathrm{d}x}{1+x^4},$$

where we have set $|\xi|^{1/2}\phi = x$. However, in two dimensions the corresponding integral is: $\int_0^\pi \frac{\mathrm{d}\phi}{1+|\xi|^2(1-\cos\phi)^2}$. For $\phi \simeq 0$ we now have

$$\frac{\mathrm{d}\phi}{1+|\xi|^2(1-\cos\phi)^2} \simeq \frac{\mathrm{d}\phi}{1+|\xi|^2\phi^4} = \frac{1}{|\xi|^{1/2}}\frac{\mathrm{d}x}{1+x^4}.$$

This explains the "loss" of $1/4$ derivatives in L^2. However, away from the light cone, ρ_S does have as many derivatives in L^2 as ρ_B. This is the content of the next theorem.

THEOREM 2. – *Consider the kinetic-transport equation* (2) *in two dimensions. Then, the averages over spheres* (1) *satisfy, additionally to the Sobolev regularity* (5)*, the hyperbolic Sobolev regularity*

$$(21) \qquad \bigl\|w_+^{1/4}w_-^{1/4}\widehat{\rho_S}\bigr\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \leqslant C\,\bigl(\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^2\times S^1)} + \|g\|_{L^2(\mathbb{R}\times\mathbb{R}^2\times S^1)}\bigr).$$

Proof. – We shall actually prove the following pointwise estimate:

$$w_+^{1/4}(\tau,\xi)w_-^{1/4}(\tau,\xi)\bigl|\widehat{\rho_S}(\tau,\xi)\bigr| \leqslant C\bigl(\bigl\|\hat{f}(\tau,\xi,\cdot)\bigr\|_{L^2(S^1)} + \bigl\|\hat{g}(\tau,\xi,\cdot)\bigr\|_{L^2(S^1)}\bigr).$$

Since

$$w_+^{1/4}w_-^{1/4} \leqslant C\bigl(1+\dot{w}_+^{1/4}+\dot{w}_+^{1/4}\dot{w}_-^{1/4}\bigr)$$

it suffices to prove the following estimates:

$$\left|\widehat{\rho_S}(\tau,\xi)\right| \leqslant C\left\|\hat{f}(\tau,\xi,\cdot)\right\|_{L^2(S^1)}, \tag{22}$$

$$\dot{w}_+^{1/4}(\tau,\xi)\left|\widehat{\rho_S}(\tau,\xi)\right| \leqslant C\left(\left\|\hat{f}(\tau,\xi,\cdot)\right\|_{L^2(S^1)} + \left\|\hat{g}(\tau,\xi,\cdot)\right\|_{L^2(S^1)}\right), \tag{23}$$

$$\dot{w}_+^{1/4}(\tau,\xi)\dot{w}_-^{1/4}(\tau,\xi)\left|\widehat{\rho_S}(\tau,\xi)\right| \leqslant C\left(\left\|\hat{f}(\tau,\xi,\cdot)\right\|_{L^2(S^1)} + \left\|\hat{g}(\tau,\xi,\cdot)\right\|_{L^2(S^1)}\right). \tag{24}$$

The first estimate follows trivially from the definition (1) of ρ_S. Observe also that for (τ,ξ) such that $|\tau|+|\xi| \leqslant 1$ we have $\dot{w}_+(\tau,\xi) \leqslant 1$ and $\dot{w}_-(\tau,\xi) \leqslant 1$, therefore, in this case, (23) and (24) follow from (22).

It remains to prove (23) and (24) in the case $|\tau|+|\xi| \geqslant 1$. We may assume $\tau > 0$. Add f to both sides of (2) and take the Fourier transform in (τ,ξ) to get

$$\widehat{\rho_S}(\tau,\xi) = \int_{S^1} \frac{\hat{h}(\tau,\xi,v)}{1+\mathrm{i}(\tau+\xi\cdot v)}\,\mathrm{d}\sigma(v), \tag{25}$$

where $h = f+g$. Apply the Cauchy–Schwarz inequality to get

$$\left|\widehat{\rho_S}(\tau,\xi)\right| \leqslant J(\tau,\xi)\left(\int_{S^1} \left|\hat{h}(\tau,\xi,v)\right|^2\,\mathrm{d}\sigma(v)\right)^{1/2}, \tag{26}$$

where

$$J(\tau,\xi) = \left(\int_{S^1} \frac{1}{1+(\tau+\xi\cdot v)^2}\,\mathrm{d}\sigma(v)\right)^{1/2}. \tag{27}$$

Therefore it suffices to prove the following two estimates:

$$J(\tau,\xi) \leqslant \frac{C}{(|\tau|+|\xi|)^{1/4}}, \tag{28}$$

$$J(\tau,\xi) \leqslant \frac{C}{(|\tau|+|\xi|)^{1/4}\,||\tau|-|\xi||^{1/4}}. \tag{29}$$

We may again choose a v basis such that $\xi = (|\xi|,0)$. We then write $v = (\cos\theta, \sin\theta)$ to get

$$J^2(\tau,\xi) = 2\int_0^\pi \frac{1}{1+(\tau+|\xi|\cos\theta)^2}\,\mathrm{d}\theta. \tag{30}$$

The proof of (28) and (29) is most difficult for that part of the integral $\int_0^\pi \frac{\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2}$ where θ is close to π. Thus we write

$$J^2(\tau,\xi) = J_1^2(\tau,\xi) + J_2^2(\tau,\xi), \tag{31}$$

where

$$J_1^2(\tau,\xi) = 2\int_0^{2\pi/3} \frac{1}{1+(\tau+|\xi|\cos\theta)^2}\,\mathrm{d}\theta, \tag{32}$$

$$J_2^2(\tau,\xi) = 2\int_{2\pi/3}^{\pi} \frac{1}{1+(\tau+|\xi|\cos\theta)^2}\,\mathrm{d}\theta, \tag{33}$$

and estimate $J_2(\tau,\xi)$ first. Estimating $J_2(\tau,\xi)$ is most difficult when (τ,ξ) is close to the light cone $\tau=|\xi|$. Let us first consider the case $|\xi|\leqslant\tau\leqslant 2|\xi|$. We let $\phi=\pi-\theta$ to rewrite $J_2(\tau,\xi)$ as

$$J_2^2(\tau,\xi) = \int_0^{\pi/3} \frac{1}{1+(\tau-|\xi|\cos\phi)^2}\,\mathrm{d}\phi.$$

Observe that

$$\tau-|\xi|\cos\phi \geqslant |\xi|-|\xi|\cos\phi \geqslant c|\xi|\phi^2.$$

Using this, and then changing variables $\phi\to y:=\sqrt{c|\xi|}\phi$, we get

$$J_2^2(\tau,\xi) \leqslant 2\int_0^{\pi/3} \frac{\mathrm{d}\phi}{1+c^2|\xi|^2\phi^4} = \frac{2}{\sqrt{c|\xi|}}\int_0^{\sqrt{c|\xi|}\frac{\pi}{3}} \frac{\mathrm{d}y}{1+y^4} \leqslant \frac{C'}{|\xi|^{1/2}},$$

and the corresponding part of (28) is proved. On the other hand, letting $\alpha=\frac{\tau}{|\xi|}$ and observing that $\alpha\in[1,2]$, we have:

$$\begin{aligned}
\bigl(\tau-|\xi|\cos\phi\bigr)^2 &= |\xi|^2(\alpha-\cos\phi)^2\\
&= |\xi|^2\bigl((\alpha-1)+(1-\cos\phi)\bigr)^2\\
&\geqslant |\xi|^2(\alpha-1)(1-\cos\phi)\\
&\geqslant c'|\xi|\bigl(\tau-|\xi|\bigr)\phi^2\\
&\geqslant c\bigl(\tau+|\xi|\bigr)\bigl(\tau-|\xi|\bigr)\phi^2.
\end{aligned}$$

Using this and then changing variables $\phi\to x:=c^{1/2}(\tau+|\xi|)^{1/2}(\tau-|\xi|)^{1/2}\phi$ we get

$$\begin{aligned}
J_2^2(\tau,\xi) &\leqslant \int_0^{\pi/3} \frac{\mathrm{d}\phi}{1+c(\tau+|\xi|)(\tau-|\xi|)\phi^2}\\
&= \frac{1}{c^{1/2}(\tau+|\xi|)^{1/2}(\tau-|\xi|)^{1/2}} \int_0^{\frac{\pi}{3}\sqrt{c(\tau^2-|\xi|^2)}} \frac{\mathrm{d}x}{1+x^4}\\
&\leqslant \frac{C}{(\tau+|\xi|)^{1/2}(\tau-|\xi|)^{1/2}}.
\end{aligned}$$

Hence, we have shown that $J_2(\tau,\xi)$ satisfies both of the required estimates in the case $|\xi|\leqslant\tau\leqslant 2|\xi|$.

Next we consider the case $|\xi|/4<\tau<|\xi|$. We perform the change of variables $\phi\to x:=\tau-|\xi|\cos\phi$ to get

$$J_2^2(\tau,\xi) = \int_{-(|\xi|-\tau)}^{\tau-|\xi|/2} \frac{\mathrm{d}x}{(1+x^2)(x+|\xi|-\tau)^{1/2}(|\xi|+\tau-x)^{1/2}}.$$

Observe that $|\xi| + \tau - x \geqslant 3|\xi|/2$ therefore

$$J_2^2(\tau,\xi) \leqslant \frac{C}{|\xi|^{1/2}} B(\tau,\xi) \leqslant \frac{C}{(|\xi|+\tau)^{1/2}} B(\tau,\xi), \tag{34}$$

where

$$B(\tau,\xi) = \int\limits_{-(|\xi|-\tau)}^{\tau-|\xi|/2} \frac{\mathrm{d}x}{(1+x^2)(x+|\xi|-\tau)^{1/2}}.$$

Therefore, in order to prove the estimate $J_2^2(\tau,\xi) \leqslant C/(|\xi|+\tau)^{1/2}$, it suffices to show that $B(\tau,\xi) \leqslant C$. We write $B(\tau,\xi)$ as

$$\begin{aligned} B(\tau,\xi) &= \int\limits_{-(|\xi|-\tau)}^{0} \frac{\mathrm{d}x}{(1+x^2)(x+|\xi|-\tau)^{1/2}} + \int\limits_{0}^{\tau-|\xi|/2} \frac{\mathrm{d}x}{(1+x^2)(x+|\xi|-\tau)^{1/2}} \\ &=: B_1(\tau,\xi) + B_2(\tau,\xi). \end{aligned} \tag{35}$$

For $B_1(\tau,\xi)$ we have directly

$$B_1(\tau,\xi) = \int\limits_{0}^{|\xi|-\tau} \frac{\mathrm{d}y}{(1+y^2)(|\xi|-\tau-y)^{1/2}} \leqslant C.$$

For $B_2(\tau,\xi)$ use $|\xi| - \tau \geqslant 0$, to get

$$B_2(\tau,\xi) \leqslant C \int\limits_{0}^{+\infty} \frac{\mathrm{d}x}{(1+x^2)\sqrt{x}} \leqslant C.$$

We have shown that $B(\tau,\xi) \leqslant C$. This implies that

$$J_2^2(\tau,\xi) \leqslant \frac{C}{(|\xi|+\tau)^{1/2}}.$$

On the other hand, letting $\alpha = \tau/|\xi|$, and observing that $\alpha \in (0, 1/4)$ can be written as $\alpha = \cos\phi_0$, we have

$$|\alpha - \cos\phi| = |\cos\phi_0 - \cos\phi| \geqslant c(\sin\phi_0)|\phi - \phi_0|.$$

Using this estimate and then changing variables $\phi \to x := c|\xi|(\sin\phi_0)(\phi - \phi_0)$ we have

$$\begin{aligned} J_2^2(\tau,\xi) &\leqslant \int\limits_{0}^{\pi/3} \frac{\mathrm{d}\phi}{1 + |\xi|^2 c^2 (\sin\phi_0)^2 |\phi-\phi_0|^2} \leqslant \frac{1}{c|\xi|\sin\phi_0} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^2} \\ &\leqslant \frac{C}{(|\tau|+|\xi|)^{1/2} ||\tau|-|\xi||^{1/2}}, \end{aligned}$$

where we have used: $|\xi|\sin\phi_0 = (|\tau|+|\xi|)^{1/2} ||\tau|-|\xi||^{1/2}$.

We have proved the desired estimates for $J_2(\tau,\xi)$ in the case that (τ,ξ) is close to the light cone. The remaining two cases are much easier. Actually, when (τ,ξ) is away from the light cone, much better estimates can be proven. Consider first the case $\tau \geqslant 2|\xi|$. Observe that

$$\tau - |\xi|\cos\phi \geqslant \tau - |\xi| \geqslant \frac{\tau}{2}$$

therefore

$$J_2^2(\tau,\xi) = \int_0^{\pi/3} \frac{\mathrm{d}\phi}{1+(\tau-|\xi|\cos\phi)^2} \leqslant \frac{C}{\tau^2} \leqslant \begin{cases} \dfrac{C}{(\tau+|\xi|)^{1/2}}, \\ \dfrac{C}{(\tau+|\xi|)^{1/2}(\tau-|\xi|)^{1/2}}. \end{cases}$$

Finally, if $4\tau \leqslant |\xi|$, then (recall $\phi \in [0,\pi/3]$)

$$|\xi|\cos\phi - \tau \geqslant \frac{|\xi|}{2} - \tau \geqslant \frac{|\xi|}{4},$$

therefore

$$J_2^2(\tau,\xi) = \int_0^{\pi/3} \frac{\mathrm{d}\phi}{1+(\tau-|\xi|\cos\phi)^2} \leqslant \frac{C}{|\xi|^2} \leqslant \begin{cases} \dfrac{C}{(\tau+|\xi|)^{1/2}}, \\ \dfrac{C}{(\tau+|\xi|)^{1/2}(|\xi|-\tau)^{1/2}}. \end{cases}$$

We now come to the estimates for $J_1^2(\tau,\xi) = 2\int_0^{2\pi/3} \frac{\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2}$. Because θ is away from π estimating $J_1(\tau,\xi)$ is much easier. Indeed we have

$$\begin{aligned} J_1^2(\tau,\xi) &= 2\int_0^{2\pi/3} \frac{\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2} \\ &= 2\int_0^{\pi/3} \frac{\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2} + 2\int_{\pi/3}^{2\pi/3} \frac{\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2}. \end{aligned}$$

For $\theta \in (0,\pi/3)$ we have $\tau + |\xi|\cos\theta > \tau + |\xi|/2$ therefore

$$\int_0^{2\pi/3} \frac{\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2} \leqslant \frac{C}{(\tau+|\xi|)^2} \leqslant \begin{cases} \dfrac{C}{(\tau+|\xi|)^{1/2}}, \\ \dfrac{C}{(\tau+|\xi|)^{1/2}|\tau-|\xi||^{1/2}}. \end{cases}$$

For $\theta \in (\pi/3\ ,\ 2\pi/3)$ we have $\sin\theta \geqslant \sqrt{3}/2$, therefore:

$$\int_{\pi/3}^{2\pi/3} \frac{\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2} \leqslant \frac{2}{\sqrt{3}} \int_{\pi/3}^{2\pi/3} \frac{\sin\theta\,\mathrm{d}\theta}{1+(\tau+|\xi|\cos\theta)^2} \leqslant \frac{C}{\tau+|\xi|}$$
$$\leqslant \begin{cases} \dfrac{C}{(\tau+|\xi|)^{1/2}}, \\ \dfrac{C}{(\tau+|\xi|)^{1/2}|\tau-|\xi||^{1/2}}, \end{cases}$$

where we have used (17).

This completes the proof of the theorem. □

COROLLARY 1 (Strichartz-type estimate). – *Under the assumptions of the last theorem the following estimate holds true*:

$$\|\rho_S\|_{L^3(\mathbb{R}^{1+2})} \leqslant C\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^1)} + \|g\|_{L^2(\mathbb{R}^{1+2}\times S^1)}\big). \tag{36}$$

Proof. – It is an immediate consequence of (21) and the following embedding result of [26]:

$$\|u\|_{L^3(\mathbb{R}^3)} \leqslant C\big\|\big(1+\big||\tau|-|\xi|\big|\big)^{1/4}\big(1+|\tau|+|\xi|\big)^{1/4}\hat{u}(\tau,\xi)\big\|_{L^2(\mathbb{R}^3)}. \qquad \square \tag{37}$$

Remarks. – (1) The remarks of Section 2 apply here too. In particular estimate (21) implies the classical averaging lemma (4).

(2) The classical averaging result $\rho_S \in H^{1/4}$, together with Sobolev embedding would only prove $\rho_S \in L^{12/5}$.

4. Averages on spheres for $\partial_t f + v\cdot\nabla_x f = \partial_v g$

In this section we consider equations of the form:

$$\partial_t f + v\cdot\nabla_x f = \partial_v g. \tag{38}$$

We define (with a slight abuse of notations), for a given function $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$\rho_B(t,x) = \int_{|v|\leqslant 1} f(t,x,v)\psi(v)\,\mathrm{d}v, \tag{39}$$

$$\rho_S(t,x) = \int_{S^{d-1}} f(t,x,v)\,\mathrm{d}\sigma(v). \tag{40}$$

The classical averaging lemma in [8] provides the following estimate for ρ_B:

$$\|\rho_B\|_{H^{1/4}(\mathbb{R}^{1+d})} \leqslant C(d)\big(\|f\|_{L^2(\mathbb{R}^{1+2d})} + \|g\|_{L^2(\mathbb{R}^{1+2d})}\big). \tag{41}$$

The crucial step in the proof of (41) is an integration by parts which removes the v-derivative from g. The presence of the cut-off function ψ guarantees that no boundary terms will come up.

In order to be able to integrate by parts on the sphere we consider v-derivatives which are tangential to the sphere. More precisely we define ([25], pp. 51–53):

$$\Omega_v^{ij} = v_i\frac{\partial}{\partial v_j} - v_j\frac{\partial}{\partial v_i}, \quad 1\leqslant i,j\leqslant d. \tag{42}$$

Then, for any smooth functions $a, b : S^{d-1} \to \mathbb{R}$, we have

$$\int_{S^{d-1}} \big(\Omega_v^{ij} a(v)\big) b(v)\, \mathrm{d}\sigma(v) = - \int_{S^{d-1}} a(v) \big(\Omega_v^{ij} b(v)\big)\, \mathrm{d}\sigma(v). \tag{43}$$

Because there are no boundary terms in (43) we do not need to introduce a cut-off function $\psi(v)$ in the definition (40) of ρ_S.

THEOREM 3. – *Consider the equation*

$$\partial_t f + v \cdot \nabla_x f = \Omega_v^{ij} g, \tag{44}$$

where Ω_v^{ij} *is as in* (42)*, and define* ρ_S *as in* (40). *Then*:
for $d \geqslant 3$

$$\|\rho_S\|_{H^{1/4}(R^{1+d})} \leqslant C(d)\big(\|f\|_{L^2(\mathbb{R}^{1+d}\times S^{d-1})} + \|g\|_{L^2(\mathbb{R}^{1+d}\times S^{d-1})}\big), \tag{45}$$

for $d = 2$

$$\|\rho_S\|_{H^{1/8}(R^{1+2})} \leqslant C(d)\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^{1})} + \|g\|_{L^2(\mathbb{R}^{1+2}\times S^{1})}\big). \tag{46}$$

Remark. – This theorem is obtained by the classical computation. We improve the two-dimensional result later in this section.

Proof. – We shall prove the following two pointwise estimates:
for $d \geqslant 3$:

$$w_+(\tau,\xi)^{1/4}\big|\widehat{\rho_S}(\tau,\xi)\big| \leqslant C(d)\big(\big\|\hat f(\tau,\xi,\cdot)\big\|_{L^2(S^{d-1})} + \big\|\hat g(\tau,\xi,\cdot)\big\|_{L^2(S^{d-1})}\big), \tag{47}$$

for $d = 2$:

$$w_+(\tau,\xi)^{1/8}\big|\widehat{\rho_S}(\tau,\xi)\big| \leqslant C(d)\big(\big\|\hat f(\tau,\xi,\cdot)\big\|_{L^2(S^{1})} + \big\|\hat g(\tau,\xi,\cdot)\big\|_{L^2(S^{1})}\big). \tag{48}$$

If $|\tau| + |\xi| \leqslant 1$ both estimates are immediate consequences of the definition of ρ_S. Assume therefore that $\tau + |\xi| > 1$. We now consider a $\lambda := \lambda(|\tau| + |\xi|)$ to be chosen later and we define $\tau' = \tau/\lambda$, $\xi' = \xi/\lambda$. Take the Fourier transform of (51) with respect to (t, x) and add $\lambda \hat f(\tau,\xi,v)$ to get

$$\begin{aligned}
\widehat{\rho_S}(\tau,\xi) &= \int_{S^{d-1}} \frac{\Omega_v^{ij}\hat g(\tau,\xi,v)}{\lambda + \mathrm{i}(\tau + \xi\cdot v)}\,\mathrm{d}\sigma(v) + \int_{S^{d-1}} \frac{\lambda \hat f(\tau,\xi,v)}{\lambda + \mathrm{i}(\tau+\xi\cdot v)}\,\mathrm{d}\sigma(v) \\
&= -\mathrm{i}\int_{S^{d-1}} \hat g(\tau,\xi,v)\frac{v_i\xi_j - v_j\xi_i}{(\lambda + \mathrm{i}(\tau+\xi\cdot v))^2}\,\mathrm{d}\sigma(v) + \int_{S^{d-1}} \frac{\lambda \hat f(\tau,\xi,v)}{\lambda + \mathrm{i}(\tau+\xi\cdot v)}\,\mathrm{d}\sigma(v)
\end{aligned}$$

after integration by parts using (43). Therefore, we have

$$\big|\widehat{\rho_S}(\tau,\xi)\big| \leqslant \big\|\hat g(\tau,\xi,v)\big\|_{L_v^2} J_1(\tau,\xi) + \big\|\hat f(\tau,\xi,v)\big\|_{L_v^2} J_2(\tau,\xi), \tag{49}$$

where

$$J_1(\tau,\xi) = \frac{1}{\lambda}\left(\int_{S^1} \frac{|v_1\xi_2' - v_2\xi_1'|^2}{(1 + (\tau' + \xi'\cdot v)^2)^2}\,\mathrm{d}\sigma(v)\right)^{1/2},$$

$$J_2(\tau,\xi)=\left(\int\limits_{S^1}\frac{1}{(1+(\tau'+\xi'\cdot v)^2)^2}\,\mathrm{d}\sigma(v)\right)^{1/2}.$$

Next, we can estimate as in the previous section, choosing $\lambda=(|\tau|+|\xi|)^{1/2}$,

$$\begin{aligned}
J_1(\tau,\xi)&\leqslant\left(\int\limits_{S^{d-1}}\frac{1}{(1+(\tau'+\xi'\cdot v)^2)^2}\,\mathrm{d}\sigma(v)\right)^{1/2}\frac{|\xi|}{\lambda^2}\\
&\leqslant\begin{cases}\frac{C(d)}{(|\tau'|+|\xi'|)^{1/2}}\frac{|\xi|}{\lambda^2}, & \text{for } d\geqslant 3,\\ \frac{C(d)}{(|\tau'|+|\xi'|)^{1/4}}\frac{|\xi|}{\lambda^2}, & \text{for } d=2\end{cases}\\
&\leqslant\begin{cases}\frac{C(d)}{(|\tau|+|\xi|)^{1/4}}, & \text{for } d\geqslant 3,\\ \frac{C(d)}{(|\tau|+|\xi|)^{1/8}}, & \text{for } d=2.\end{cases}
\end{aligned}$$

To estimate $J_2(\tau,\xi)$ we work as in Sections 2 and 3:

$$\begin{aligned}
J_2(\tau,\xi)&\leqslant\left(\int\limits_{S^{d-1}}\frac{1}{1+(\tau'+\xi'\cdot v)^2}\,\mathrm{d}\sigma(v)\right)^{1/2}\\
&\leqslant\begin{cases}\frac{C(d)}{(|\tau'|+|\xi'|)^{1/2}}, & \text{for } d\geqslant 3,\\ \frac{C(d)}{(|\tau'|+|\xi'|)^{1/4}}, & \text{for } d=2\end{cases}\\
&\leqslant\begin{cases}\frac{C(d)}{(|\tau|+|\xi|)^{1/4}}, & \text{for } d\geqslant 3,\\ \frac{C(d)}{(|\tau|+|\xi|)^{1/8}}, & \text{for } d=2.\end{cases}\qquad\square
\end{aligned}$$

The use of the operators Ω_v^{ij} in the right-hand side of (44) has been dictated by the need of integrating by parts on the sphere. However, their presence provides the right-hand side with a special structure. This is apparent in the term $\int_{S^{d-1}}\hat{g}(\tau,\xi,v)\frac{(v_i\xi_j-v_j\xi_i)}{(\lambda+\mathrm{i}(\tau+\xi\cdot v))^2}\,\mathrm{d}\sigma(v)$ which comes up after the integration by parts is performed. Quantities like $v_i\xi_j-v_j\xi_i$ often come up in the study of Lorentz invariant systems of nonlinear wave equations and are known there as *null forms*. They allow proving better estimates than those valid for general quadratic forms (see [18]). We shall now use this observation to prove the following theorem which improves (46) not only away from the light cone $|\xi|=\tau$ but in the whole (τ,ξ)-space.

THEOREM 4. – *Let $d=2$ and consider the equation* (44). *Then the average ρ_S defined by* (40) *satisfies*:

$$\left\|w_+(\tau,\xi)^{1/7}\widehat{\rho_S}(\tau,\xi)\right\|_{L^2(\mathbb{R}^{1+2})}\leqslant C(d)\big(\|f\|_{L^2(\mathbb{R}^{1+d}\times S^1)}+\|g\|_{L^2(\mathbb{R}^{1+d}\times S^1)}\big).\tag{50}$$

Proof. – We recall the formula (49) and choose $\lambda=|\xi|^{3/7}$. To estimate $J_1(\tau,\xi)$ we choose a v basis where $\xi=(|\xi|,0)$. We then write $v=(\cos\theta,\sin\theta)$ to get:

$$\begin{aligned}
J_1(\tau,\xi)&=\frac{1}{\lambda^2}\left(2\int\limits_0^\pi\frac{|\xi|^2\sin^2\theta}{(1+(\tau'+|\xi'|\cos\theta)^2)^2}\,\mathrm{d}\theta\right)^{1/2}\\
&\leqslant C\frac{|\xi|}{\lambda^2}\left(\int\limits_0^\pi\frac{\sin\theta}{1+(\tau'+|\xi'|\cos\theta)^2}\,\mathrm{d}\theta\right)^{1/2}
\end{aligned}$$

$$\leqslant \frac{|\xi|}{\lambda^2}\,\frac{C}{(|\tau'|+|\xi'|)^{1/2}}$$
$$\leqslant \frac{C}{(|\tau|+|\xi|)^{1/7}},$$

where we have used (17).

For $J_2(\tau,\xi)$ we have, working as in Section 3,

$$J_2(\tau,\xi) \leqslant \left(2\int_0^{\pi} \frac{1}{1+(\tau'+|\xi'|\cos\theta)^2}\,\mathrm{d}\theta\right)^{1/2}$$
$$\leqslant \frac{C}{(|\tau|'+|\xi'|)^{1/4}}$$
$$\leqslant \frac{C}{(|\tau|+|\xi|)^{1/7}}.$$

We are done in the case $|\tau|+|\xi|>1$. The result in the remaining case $|\tau|+|\xi|\leqslant 1$ is immediate from the definition of ρ_S. □

Again, we can see that the average over the sphere in two dimensions does not satisfy the same estimate as the average over the ball. However, as in Section 3, we can prove an estimate showing that away from the light cone $|\xi|=|\tau|$, ρ_S does have as many derivatives in L^2 as ρ_B (1/4 in this case).

THEOREM 5. – *Let $d=2$ and consider the equation* (44). *Then the average ρ_S defined by* (40) *satisfies*:

$$\left\| w_-^{3/16} w_+^{1/16} \widehat{\rho_S} \right\|_{L^2(\mathbb{R}^{1+2})} \leqslant C(d)\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^1)} + \|g\|_{L^2(\mathbb{R}^{1+2}\times S^1)}\big), \tag{51}$$

and the Strichartz type inequality

$$\|\rho_S\|_{L_t^3 L_x^{24/11}(\mathbb{R}^{1+2})} \leqslant C(d)\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^1)} + \|g\|_{L^2(\mathbb{R}^{1+2}\times S^1)}\big), \tag{52}$$

where the mixed space time norm is defined as

$$\left\| f(t,x)\right\|_{L_t^p L_x^q(\mathbb{R}^{1+d})} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \left|f(t,x)\right|^q \mathrm{d}x\right)^{p/q} \mathrm{d}t\right)^{1/p}.$$

Remark. – Combining (51) and (50), we get:

$$\left\| w_-(\tau,\xi)^{\delta} w_+(\tau,\xi)^{s} \widehat{\rho_S}(\tau,\xi)\right\|_{L^2(\mathbb{R}^{1+2})} \leqslant C\big(\|f\|_{L^2(\mathbb{R}^{1+2}\times S^1)} + \|g\|_{L^2(\mathbb{R}^{1+2}\times S^1)}\big)$$

for $s=\frac{1-\theta}{7}+\frac{\theta}{16}$, $\delta=\frac{3\theta}{16}$ and $0\leqslant\theta\leqslant 1$.

Proof. – Working exactly as in the proof of the last theorem for J_1 and in Section 3 for J_2, we get:

$$J_1(\tau,\xi) \leqslant \frac{C}{w_+(\tau',\xi')^{1/2}}\,\frac{|\xi|}{\lambda^2},$$

$$J_2(\tau,\xi) \leqslant \frac{C}{w_-(\tau',\xi')^{1/4} w_+(\tau',\xi')^{1/4}}.$$

Choosing $\lambda^4 = w_-(\tau,\xi)^{1/2} w_+(\tau,\xi)^{3/2}$, we get the following estimate for both J_1 and J_2:

$$J_i(\tau,\xi) \leqslant \frac{C}{w_-(\tau,\xi)^{3/16} w_+(\tau,\xi)^{1/16}},$$

and the result follows. To prove (52) use (51) and the following embedding result of [26]:

$$\|u\|_{L_t^3 L_x^{24/11}(\mathbb{R}^{1+2})} \leqslant C \left\| w_-(\tau,\xi)^{3/16} w_+(\tau,\xi)^{1/16} \hat{u}(\tau,\xi) \right\|_{L^2(\mathbb{R}^{1+2})}. \qquad \square$$

5. Averages on spheres for the initial value problem

In this section we consider averages over spheres for the homogeneous initial value problem. Averages over balls have been studied in [15], [3] and here we follow the presentation in [2]. Then, it is possible to show additional regularizing effects in time from L^∞ to L^2. We denote by $\tilde{}$ the Fourier transform in space-time, i.e., $(t,x) \to (\tau,\xi)$ and by $\hat{}$ the Fourier transform in space, i.e., $x \to \xi$.

THEOREM 6. – *Consider the initial value problem in three dimensions*:

$$\partial_t f + v \cdot \nabla_x f = 0, \qquad f(0,x,v) = f_0(x,v). \tag{53}$$

Then, the averages on spheres (1) *satisfy*

$$\left\| \big(|\tau| + |\xi|\big)^{1/2} \tilde{\rho_S}(\tau,\xi) \right\|_{L^2(\mathbb{R}^{1+3})} \leqslant C \left\| f_0(x,v) \right\|_{L^2(\mathbb{R}^3 \times S^2)}. \tag{54}$$

Proof. – We have

$$\tilde{\rho_S}(\tau,\xi) = \int_{S^2} \widehat{f_0}(\xi,v) \delta(\tau + \xi \cdot v)\, \mathrm{d}\sigma(v).$$

We shall prove that for any ξ,

$$\int_{\mathbb{R}} \big(|\tau| + |\xi|\big) \left|\widetilde{\rho_S}(\tau,\xi)\right|^2 d\tau \leqslant C \int_{S^2} \left|\widehat{f_0}(\xi,v)\right|^2 \mathrm{d}\sigma(v). \tag{55}$$

The desired result follows from (55) by integrating in ξ. To prove (55) we may choose a v basis where $\xi = (0,0,|\xi|)$. Then, using spherical coordinates, we get:

$$\tilde{\rho_S}(\tau,\xi) = \int_0^{2\pi} \int_0^{\pi} \widehat{f_0}(\xi, \cos\phi\ \sin\theta, \sin\phi\ \sin\theta, \cos\theta) \delta\big(\tau + |\xi|\cos\theta\big) \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi.$$

Observe that for $|\tau| > |\xi|$ we have $\tau + |\xi|\cos\theta \geqslant |\tau| - |\xi| > 0$ therefore $\widetilde{\rho_S}(\tau,\xi) = 0$. On the other hand, for $|\tau| < |\xi|$, we have $|\tau| + |\xi| \leqslant 2|\xi|$. Therefore, to prove (55), it suffices to prove:

$$\int_{-|\xi|}^{|\xi|} |\xi| \left|\tilde{\rho}(\tau,\xi)\right|^2 \mathrm{d}\tau \leqslant C \int_{S^2} \left|\widehat{f_0}(\xi,v)\right|^2 \mathrm{d}\sigma(v). \tag{56}$$

Fix τ, ξ, ϕ and consider

$$\int_0^\pi \widehat{f_0}(\xi, \cos\phi \ \sin\theta, \sin\phi \ \sin\theta, \cos\theta)\delta\big(\tau + |\xi|\cos\theta\big)\sin\theta\,\mathrm{d}\theta.$$

Change variables $\theta \to x := \tau + |\xi|\cos\theta$, to get:

$$\begin{aligned}
&\int_0^\pi \widehat{f_0}(\xi, \cos\phi \ \sin\theta, \sin\phi \ \sin\theta, \cos \ \theta)\delta\big(\tau + |\xi|\cos\theta\big)\sin\theta\,\mathrm{d}\theta \\
&\quad = \int_{\tau-|\xi|}^{\tau+|\xi|} \widehat{f_0}\left(\xi, \left(1 - \left(\frac{x-\tau}{|\xi|}\right)^2\right)^{1/2}\cos\phi, \left(1 - \left(\frac{x-\tau}{|\xi|}\right)^2\right)^{1/2}\sin\phi, \frac{x-\tau}{|\xi|}\right)\delta(x)\ \frac{\mathrm{d}x}{|\xi|} \\
&\quad = \frac{1}{|\xi|}\ \widehat{f_0}\left(\xi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\cos\phi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\sin\phi, -\frac{\tau}{|\xi|}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&|\xi|\tilde{\rho_S}(\tau,\xi) \\
&\quad = \int_0^{2\pi} \widehat{f_0}\left(\xi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\cos\phi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\sin\phi, -\frac{\tau}{|\xi|}\right)\mathrm{d}\phi.
\end{aligned}$$

Using Cauchy–Schwarz inequality, we get:

$$\begin{aligned}
&|\xi|^2\big|\tilde{\rho_S}(\tau,\xi)\big|^2 \\
&\quad \leqslant C\int_0^{2\pi} \left|\widehat{f_0}\left(\xi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\cos\phi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\sin\phi, -\frac{\tau}{|\xi|}\right)\right|^2\mathrm{d}\phi.
\end{aligned}$$

For fixed ξ, integrate in τ to get:

$$\begin{aligned}
&\int_{-|\xi|}^{|\xi|} |\xi|^2\big|\tilde{\rho_S}(\tau,\xi)\big|^2\,\mathrm{d}\tau \\
&\quad \leqslant C\int_{-|\xi|}^{|\xi|}\int_0^{2\pi} \left|\widehat{f_0}\left(\xi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\cos\phi, \left(1 - \left(\frac{\tau}{|\xi|}\right)^2\right)^{1/2}\sin\phi, -\frac{\tau}{|\xi|}\right)\right|^2\mathrm{d}\phi\,\mathrm{d}\tau.
\end{aligned}$$

Change variables $\tau \to \theta$, where $\cos\theta = -\tau/|\xi|$, to get:

$$\begin{aligned}
\int_{-|\xi|}^{|\xi|} |\xi|\big|\tilde{\rho_S}(\tau,\xi)\big|^2\,\mathrm{d}\tau &\leqslant C\int_0^\pi\int_0^{2\pi}\big|\widehat{f_0}(\xi, \cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)\big|^2\ \sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= C\int_{S^2}\big|\widehat{f_0}(\xi, v)\big|^2\,\mathrm{d}\sigma(v).
\end{aligned}$$

This proves (56). Integrate in ξ to get the desired result. □

Again, in the case of two space dimensions the above result does not hold true and a half derivative is lost (we do not prove this fact which follows the spirit of the previous sections). We can recover the full $1/2$ deivative as follows.

THEOREM 7 (Averaging and Strichartz inequality for the i.v.p.). – *Consider the initial value problem* (53) *in two dimensions. Then, the averages on spheres* (1) *satisfy*:

$$\left\| \dot{w}_+(\tau,\xi)^{1/4}\dot{w}_-(\tau,\xi)^{1/4}\tilde{\rho}_S(\tau,\xi)\right\|_{L^2(\mathbb{R}^{1+2})} \leqslant C\left\| f_0(x,v)\right\|_{L^2(\mathbb{R}^2\times S^1)}, \tag{57}$$

$$\|\rho_S\|_{L^3(\mathbb{R}^{1+2})} \leqslant C\left\| f_0(x,v)\right\|_{L^2(\mathbb{R}^2\times S^1)}. \tag{58}$$

Proof. – We have

$$\tilde{\rho}_S(\tau,\xi) = \int_{S^1} \widehat{f_0}(\xi,v)\delta(\tau+\xi\cdot v)\,\mathrm{d}v.$$

We shall prove that for any ξ we have:

$$\int_{\mathbb{R}} \big||\tau|-|\xi|\big|^{1/2}\big(|\tau|+|\xi|\big)^{1/2}\big|\widetilde{\rho_S}(\tau,\xi)\big|^2\,\mathrm{d}\tau \leqslant C\int_{S^1}\big|\widehat{f_0}(\xi,v)\big|^2\,\mathrm{d}\sigma(v). \tag{59}$$

To prove (59) we may assume $\xi = (|\xi|, 0)$. Then, using polar coordinates, we have:

$$\tilde{\rho}_S(\tau,\xi) =: \tilde{\rho}_1(\tau,\xi) + \tilde{\rho}_2(\tau,\xi),$$

where

$$\tilde{\rho}_1(\tau,\xi) = \int_0^{\pi} \widehat{f_0}(\xi,\cos\phi,\sin\phi)\delta\big(\tau+|\xi|\cos\phi\big)\,\mathrm{d}\phi,$$

$$\tilde{\rho}_2(\tau,\xi) = \int_{\pi}^{2\pi} \widehat{f_0}(\xi,\cos\phi,\sin\phi)\delta\big(\tau+|\xi|\cos\phi\big)\,\mathrm{d}\phi.$$

We shall estimate ρ_1, the proof for ρ_2 being similar. Observe that if $|\tau| > |\xi|$ then $\tau + |\xi|\cos\phi \geqslant |\tau|-|\xi| > 0$. Therefore $\tilde{\rho}_1(\tau,\xi) = 0$. Now assume that $|\tau| < |\xi|$. We change variables $\phi \to x := \tau + |\xi|\cos\phi$ to get

$$\begin{aligned}\tilde{\rho}_1(\tau,\xi) &= \int_{\tau-|\xi|}^{|\tau|+|\xi|} \widehat{f_0}\left(\xi, \frac{x-\tau}{|\xi|}, \left(1-\left(\frac{x-\tau}{|\xi|}\right)^2\right)^{1/2}\right)\frac{\delta(x)\,\mathrm{d}x}{|\xi|\big(1-\big(\frac{x-\tau}{|\xi|}\big)^2\big)^{1/2}}\\ &= \widehat{f_0}\left(\xi,-\frac{\tau}{|\xi|},\left(1-\frac{\tau^2}{|\xi|^2}\right)^{1/2}\right)\frac{1}{|\xi|\big(1-\frac{\tau^2}{|\xi|^2}\big)^{1/2}}.\end{aligned}$$

Hence

$$\big(|\xi|-|\tau|\big)^{1/2}\big(|\xi|+|\tau|\big)^{1/2}\big|\tilde{\rho}_1(\tau,\xi)\big|^2 = \frac{\big|\widehat{f_0}\big(\xi,-\frac{\tau}{|\xi|},\big(1-\frac{\tau^2}{|\xi|^2}\big)^{1/2}\big)\big|^2}{|\xi|\big(1-\frac{\tau^2}{|\xi|^2}\big)^{1/2}}.$$

For fixed ξ, we integrate in τ and then perform the change variables $\tau \to \phi$, where $\cos\phi = -\tau/|\xi|$, and we obtain

$$\int_{-|\xi|}^{+|\xi|} \big(|\xi| - |\tau|\big)^{1/2}\big(|\xi| + |\tau|\big)^{1/2}\big|\tilde{\rho}_1(\tau,\xi)\big|^2\,\mathrm{d}\tau = \int_0^{\pi} \big|\widehat{f_0}(\xi,\cos\phi,\sin\phi)\big|^2\,\mathrm{d}\phi$$
$$\leqslant \int_{S^1} \big|\widehat{f_0}(\xi,v)\big|^2\,\mathrm{d}\sigma(v).$$

This proves (59). Integrating in ξ, we obtain the desired result. Finally (58) is proven using (57) and the embedding (37). □

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